

RATIONAL HOMOTOPY OF MAPS BETWEEN CERTAIN COMPLEX GRASSMANN MANIFOLDS

PRATEEP CHAKRABORTY* AND SHREEDEVI K. MASUTI**

ABSTRACT. Let $G_{n,k}$ denote the complex Grassmann manifold of k -dimensional vector subspaces of \mathbb{C}^n . Assume $l, k \leq \lfloor n/2 \rfloor$. We show that, for sufficiently large n , any continuous map $h : G_{n,l} \rightarrow G_{n,k}$ is rationally null homotopic if (i) $1 \leq k < l$, (ii) $2 < l < k < 2(l-1)$, (iii) $1 < l < k$, l divides n but l does not divide k .

1. INTRODUCTION

An important problem in homotopy theory is to classify, up to homotopy, maps between two given topological spaces. This is often a difficult problem even when the spaces involved are well-behaved spaces such as simply connected compact smooth manifolds. The determination of the set of *rational* homotopy classes of maps between such spaces is more tractable, thanks to the work of Sullivan [20]. In the special case when the spaces involved are *rationally formal* in the sense of rational homotopy theory this problem is reduced to studying the graded algebra homomorphisms between their rational (singular) cohomology algebras. See [5, p.156].

Denote by $G_{n,k}$ the complex Grassmann manifold of k -dimensional vector subspaces of \mathbb{C}^n . It can be identified with the homogeneous space $U(n)/(U(k) \times U(n-k))$ where $U(n)$ denotes the group of $n \times n$ unitary matrices. In particular it is a simply connected Hermitian symmetric space of complex dimension $k(n-k)$. Thus $G_{n,k}$ is a rationally formal space [5, p.162]. Our aim in this paper is to establish the following result. Since $G_{n,k} \cong G_{n,n-k}$, we assume that $1 \leq k \leq \lfloor n/2 \rfloor$.

Theorem 1.1. *Let $l, k \leq \lfloor n/2 \rfloor$. (i) Suppose that $n \geq 2l^2 + l - 2$, $1 \leq k < l$. Then any continuous map $h : G_{n,l} \rightarrow G_{n,k}$ is rationally null homotopic.*

(ii) Suppose that $n \geq 3k^2 - 2$. Then any continuous map $h : G_{n,l} \rightarrow G_{n,k}$ is rationally null homotopic in any of the following cases: (a) $2 < l < k < 2(l-1)$, (b) $1 < l < k$, l divides n but l does not divide k .

As a consequence we see that the set $[G_{n,l}, G_{n,k}]$ of homotopy classes of maps between the indicated Grassmann manifolds is finite when n, k, l are as in the above theorem. The first part of the theorem implies that for a fixed l and $k < l$, with at most *finitely* many exceptions the set $[G_{n,l}, G_{n,k}]$ is finite. We shall in fact establish a stronger statement than Case (ii)(b) of Theorem 1.1 covering many more cases. See Remark 5.2 for the detailed discussion of this paragraph.

We now recall briefly the history of the problem of classifying maps between complex Grassmann manifolds. It was shown by Friedlander [6] that the complex Grassmann manifolds admit self-maps of arbitrarily high degree. The classification of endomorphisms of the cohomology algebras of the complex Grassmann manifolds has been studied in [16], [3], [8], [11], with applications to fixed point property. See also [12] in which self-maps of certain complex flag manifolds are considered. Study of continuous maps via their induced homomorphisms in mod 2 cohomology algebras between distinct

2010 *Mathematics Subject Classification.* Primary 55S37, 13A02; Secondary 57T15.

Key words and phrases. Grassmann manifold, homotopy class of maps, graded algebra homomorphism, cohomology algebra.

The second author thanks Indian Statistical Institute Bangalore for providing local hospitality during the course of writing this paper. She also thanks Department of Atomic Energy, Government of India, for providing funding for her post doctoral studies during which this work is done.

real Grassmann manifolds was initiated in [13]. In [18], degrees of maps between distinct oriented real Grassmann manifolds of the *same* dimension was determined. The same problem for distinct complex Grassmann manifolds was considered in [19]. We point out that, Paranjape and Srinivas [17] showed that any non-constant *algebraic morphism* between two complex Grassmann manifolds of the same dimension is an isomorphism provided that the target is *not* the complex projective space.

One could consider the more general problem of classifying continuous maps $G_{m,l} \rightarrow G_{n,k}$. Since $G_{n,k}$ is a complex flag manifold; this problem of classification, up to rational homotopy, boils down to classifying graded algebra homomorphisms between the cohomology algebras with rational coefficients $H^*(G_{n,k}; \mathbb{Q})$ and $H^*(G_{m,l}; \mathbb{Q})$ (see Theorem 5.1). Since $H^*(G_{m-1,l}; \mathbb{Q})$ is a quotient of $H^*(G_{m,l}; \mathbb{Q})$, the vanishing of any graded algebra homomorphism from $H^*(G_{n,k}; \mathbb{Q})$ to $H^*(G_{n,l}; \mathbb{Q})$ in positive degrees when $m > n$ does imply the same conclusion for graded algebra homomorphisms from $H^*(G_{n,k}; \mathbb{Q})$ to $H^*(G_{m,l}; \mathbb{Q})$. Partial results when $k < l$ and $m - l > n - k$ were obtained in [4]. The nature of the structure of the rational cohomology algebra of the complex Grassmann manifolds still makes the problem quite challenging, and perhaps explains why the problem still remains unsolved for general values of m, n, k, l .

2. PRELIMINARIES

Let $\gamma_{n,k}$ be the “tautological bundle” over $G_{n,k}$, whose fiber over a point $V \in G_{n,k}$ is the k -dimensional complex vector space V . This vector bundle $\gamma_{n,k}$ is a rank k -subbundle of the rank n trivial bundle ε^n . Let $\gamma_{n,k}^\perp$ denote the orthogonal complement of $\gamma_{n,k}$ in ε^n (with respect to a hermitian metric on \mathbb{C}^n). Let $x_i := c_i(\gamma_{n,k}) \in H^{2i}(G_{n,k}; \mathbb{Z})$ be the i -th Chern class of $\gamma_{n,k}$, $1 \leq i \leq k$. Denoting the total Chern class of a vector bundle η by $c(\eta)$ we see that $c(\gamma_{n,k})c(\gamma_{n,k}^\perp) = 1$. We write $c = 1 + x_1 + \cdots + x_k$.

From [1] and [11, Theorem 2.1], we know that the cohomology algebra of $G_{n,k}$ is

$$H^*(G_{n,k}; \mathbb{Q}) = \frac{\mathbb{Q}[x_1, \dots, x_k]}{\mathcal{I}_{n,k}},$$

where degree $x_i = 2i$ and $\mathcal{I}_{n,k}$ is the ideal generated by $(c^{-1})_{n-k+j}$, $1 \leq j \leq k$. Here $(c^{-1})_j$ denotes the homogeneous part of the formal inverse of c having (total) degree $2j$. The ideal $\mathcal{I}_{n,k}$ contains all the elements $(c^{-1})_j$, $j \geq n - k + 1$ and

$$c^{-1} = 1 + (c^{-1})_1 + (c^{-1})_2 + \cdots + (c^{-1})_{n-k} \in H^*(G_{n,k}; \mathbb{Q})$$

is in fact the total Chern class of $\gamma_{n,k}^\perp$.

We denote $(c^{-1})_{n-k+j}$, $1 \leq j \leq k$, by R_j . For $\mathbf{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$, we set $|\mathbf{n}| = n_1 + \cdots + n_k$, $\text{wt } \mathbf{n} = \sum_{i=1}^k i n_i$, $\mathbf{x}^{\mathbf{n}} = x_1^{n_1} \cdots x_k^{n_k}$ and $c_{\mathbf{n}} = \frac{|\mathbf{n}|!}{n_1! n_2! \cdots n_k!}$. By [8, p.174], we can write R_j in the following form

$$(2.1) \quad R_j = \sum_{\mathbf{n} \in \mathbb{N}^k, \text{wt } \mathbf{n} = n-k+j} (-1)^{|\mathbf{n}|} c_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}.$$

We shall use Expression (2.1) of R_j in our proofs. We recall the following well-known facts about the cohomology algebra of $G_{n,k}$, which will be used in the proofs of our results:

(1) The homogeneous polynomials $R_1, \dots, R_k \in \mathcal{I}_{n,k}$ form a *regular sequence* in the polynomial algebra $\mathbb{Q}[x_1, x_2, \dots, x_k]$, which means that $(R_1, \dots, R_k) \neq \mathbb{Q}[x_1, x_2, \dots, x_k]$, $R_1 \neq 0$ and R_{j+1} is not a zero-divisor in $\frac{\mathbb{Q}[x_1, x_2, \dots, x_k]}{(R_1, R_2, \dots, R_j)}$, for $1 \leq j \leq k-1$. See [2].

(2) The natural imbeddings $\hat{i} : G_{n,k} \hookrightarrow G_{n+1,k}$ and $\hat{j} : G_{n,k} \hookrightarrow G_{n+1,k+1}$ induce surjections $\hat{i}^* : H^*(G_{n+1,k}; \mathbb{Q}) \rightarrow H^*(G_{n,k}; \mathbb{Q})$ and $\hat{j}^* : H^*(G_{n+1,k+1}; \mathbb{Q}) \rightarrow H^*(G_{n,k}; \mathbb{Q})$, where $\hat{i}^*(x_r + \mathcal{I}_{n+1,k}) = x_r + \mathcal{I}_{n,k} = \hat{j}^*(x_r + \mathcal{I}_{n+1,k+1})$ for $1 \leq r \leq k$ and $\hat{j}^*(x_{k+1} + \mathcal{I}_{n+1,k+1}) = 0$. The homomorphism \hat{i}^* induces isomorphisms in cohomology in dimensions up to $2(n-k)$ and \hat{j}^* induces isomorphisms in cohomology in dimensions up to $2k$.

Next we shall prove a lemma, which we will need to prove the main results of this paper. Throughout this paper, by saying “a graded algebra homomorphism is trivial”, we shall mean that it vanishes in positive degrees.

Lemma 2.2. *Let $n, l, k \geq 1$ and $\mathcal{I}_{n,k} = (R_1, \dots, R_k)$. Let $\phi : \mathbb{Q}[x_1, \dots, x_k] \rightarrow \mathbb{Q}[y_1, \dots, y_l]$ be a graded algebra homomorphism such that $\phi(R_i) = 0$ for all $i = 1, \dots, k$. Then ϕ is trivial.*

Proof. Since $H^*(G_{n,k}; \mathbb{Q})$ is finite dimensional as \mathbb{Q} -vector space, there exists an integer N_i such that $x_i^{N_i} \in \mathcal{I}_{n,k}$ for all i , $1 \leq i \leq k$. Hence $(\phi(x_i))^{N_i} = \phi(x_i^{N_i}) = 0$ which implies that $\phi(x_i) = 0$ for all $i = 1, \dots, k$. Hence ϕ is trivial. \square

We recall the following proposition from [8] which will be used frequently in this paper.

Proposition 2.3. [8, Proposition 1] *Let N, m_0, n_0 be integers such that*

$$N \geq p(p-1) + n_0p + m_0(p-1).$$

Then there exist integers $m \geq m_0$ and $n \geq n_0$ such that

$$m(p-1) + np = N.$$

3. GRADED ALGEBRA HOMOMORPHISMS FROM $H^*(G_{n,k}; \mathbb{Q})$ TO $H^*(G_{n,l}; \mathbb{Q})$ FOR $k < l$

In this section we consider the possible graded algebra homomorphisms from $H^*(G_{n,k}; \mathbb{Q})$ to $H^*(G_{n,l}; \mathbb{Q})$ where $k < l$. We prove that any such graded algebra homomorphism is trivial for sufficiently large n (Theorem 3.8). In order to prove this we show that $x_{l-1}x_l$ occurs in every nonzero homogeneous polynomial of degree at most $2n$ in $\mathcal{I}_{n,l}$ for sufficiently large n (Corollary 3.6). The restriction on n is needed in the course of proving this result.

We also prove that if $P_i, 1 \leq i \leq l$, is a nonzero homogeneous polynomial in $\mathcal{I}_{n,l}$ of degree $2(n-l+i)$ then P_i contains x_{i-1}, \dots, x_l for $n \geq 2l^2 + l - 2$. To obtain these results, we first prove the following proposition. We denote the vector $(0, \dots, 1, \dots, 0) \in \mathbb{N}^l$, where 1 occurs at the i th position and 0 elsewhere, by \mathbf{e}_i . Let $1 \leq c_1, c_2, l$ be integers. For $i = 1, \dots, l$ and $\mathbf{s} \in Q(i)$, we set

$$\begin{aligned} Q(i) &:= \{\mathbf{n} \in \mathbb{N}^l \mid 1 \leq \text{wt } \mathbf{n} \leq i-1\}, \\ D_i(c_1, c_2, n, l) &:= \{x_{l-1}^{m_1} x_l^{m_2} : m_1 \geq c_1, m_2 \geq c_2 \text{ satisfying } (l-1)m_1 + lm_2 = (n-l+i)\} \text{ and} \\ D_{i,\mathbf{s}}(c_1, c_2, n, l) &:= \{\mathbf{x}^{\mathbf{s}} x_{l-1}^{m_1} x_l^{m_2} : m_1 \geq c_1, m_2 \geq c_2 \text{ satisfying } (l-1)m_1 + lm_2 = (n-l+i) - \text{wt } \mathbf{s}\}. \end{aligned}$$

We shall use D_i instead of $D_i(c_1, c_2, n, l)$ and $D_{i,\mathbf{s}}$ instead of $D_{i,\mathbf{s}}(c_1, c_2, n, l)$ to simplify notations.

Proposition 3.1. *Let $1 \leq l, c_1, c_2$ be integers such that $n-l+1 \geq l(l-1) + (c_1+l)(l-1) + c_2l$. Let $1 \leq i \leq l$ be fixed. Then*

- (1) *the sets D_i and $D_{i,\mathbf{s}}$ are non-empty;*
- (2) *every nonzero homogeneous polynomial of degree $2(n-l+i)$ in $\mathcal{I}_{n,l}$ contains at least one monomial from the set*

$$A_i := D_i \bigcup \left(\bigcup_{\mathbf{s} \in Q(i)} D_{i,\mathbf{s}} \right)$$

with nonzero coefficient.

Proof. (1): Follows from Proposition 2.3.

(2): Let $I = \mathcal{I}_{n,l}$. For fixed $1 \leq i \leq l$, let $0 \neq P_i \in I$ be a homogeneous polynomial of degree $2(n-l+i)$. Then

$$(3.2) \quad P_i = N_i R_i + \sum_{\mathbf{s} \in Q(i)} N_{i,\mathbf{s}} \mathbf{x}^{\mathbf{s}} R_{i-\text{wt } \mathbf{s}}$$

for some $N_i, N_{i,\mathbf{s}} \in \mathbb{Q}$. Suppose P_i does not contain any monomial from the set A_i .

Case 1: $i < l$

Since $n-l+i \geq n-l+1 \geq l(l-1) + (c_1+l)(l-1) + c_2 l$, by Proposition 2.3, there exist integers $m_1 \geq c_1 + l, m_2 \geq c_2$ such that

$$n-l+i = m_1(l-1) + m_2 l.$$

Observe that $\max\{j : x_j \text{ divides } \mathbf{x}^{\mathbf{s}} \text{ for some } \mathbf{s} \in Q(i)\} = i-1$. Therefore comparing the coefficients of $x_{l-1}^{m_1} x_l^{m_2}$ in (3.2), we get that $N_i = 0$.

In order to show that $N_{i,\mathbf{s}} = 0$ for all $\mathbf{s} \in Q(i)$ we order the set $Q(i)$ lexicographically and induct on \mathbf{s} . This means that $\mathbf{n} < \mathbf{m}$ in $Q(i)$ if the first nonzero entry of $\mathbf{m} - \mathbf{n}$ is positive. This defines a total order on $Q(i)$. Note that the vector $\mathbf{e}_{i-1} \in Q(i)$ is the smallest element in $Q(i)$. First we show that $N_{i,\mathbf{e}_{i-1}} = 0$. Since $n-l+1 \geq l(l-1) + (c_1+l)(l-1) + c_2 l$, again by Proposition 2.3, there exist integers $m_3 \geq c_1 + l, m_4 \geq c_2$ such that

$$n-l+1 = m_3(l-1) + m_4 l.$$

Comparing the coefficients of $x_{i-1} x_{l-1}^{m_3} x_l^{m_4}$ in (3.2), we get that $N_{i,\mathbf{e}_{i-1}} = 0$.

Let $\mathbf{s} \in Q(i)$ be fixed and assume that $N_{i,\mathbf{t}} = 0$ for $\mathbf{t} < \mathbf{s}$. Since $n-l+i - \text{wt } \mathbf{s} \geq n-l+1 \geq l(l-1) + (c_1+l)(l-1) + c_2 l$, by Proposition 2.3, there exist integers $m_5 \geq c_1 + l, m_6 \geq c_2$ such that

$$n-l+i - \text{wt } \mathbf{s} = m_5(l-1) + m_6 l.$$

Then the monomial $\mathbf{x}^{\mathbf{s}} x_{l-1}^{m_5} x_l^{m_6}$ occurs in $\mathbf{x}^{\mathbf{t}} R_{i-\text{wt } \mathbf{t}}$ implies that $\mathbf{t} \leq \mathbf{s}$. Hence comparing the coefficients of $\mathbf{x}^{\mathbf{s}} x_{l-1}^{m_5} x_l^{m_6}$ in (3.2) and using the induction hypothesis we get that $N_{i,\mathbf{s}} = 0$. Therefore $N_{i,\mathbf{s}} = 0$ for all $\mathbf{s} \in Q(i)$ which implies that $P_i = 0$, a contradiction. Hence P_i must contain at least one monomial from the set A_i .

Case 2: $i = l$

We have

$$(3.3) \quad P_l = N_l R_l + \sum_{\mathbf{s} \in Q(l)} N_{l,\mathbf{s}} \mathbf{x}^{\mathbf{s}} R_{l-\text{wt } \mathbf{s}}.$$

Since $n \geq n-l+1 \geq l(l-1) + (c_1+l)(l-1) + c_2 l$, by Proposition 2.3, there exist integers $m_7 \geq c_1 + l, m_8 \geq c_2$ such that

$$n = m_7(l-1) + m_8 l.$$

Comparing the coefficients of $x_{l-1}^{m_7} x_l^{m_8}$ in Equation (3.3), we get that

$$(-1)^{m_7+m_8} \frac{(m_7+m_8)!}{m_7! m_8!} N_l + (-1)^{m_7+m_8-1} \frac{(m_7+m_8-1)!}{(m_7-1)! m_8!} N_{l,\mathbf{e}_{l-1}} = 0.$$

Hence

$$(3.4) \quad \frac{m_7+m_8}{m_7} N_l - N_{l,\mathbf{e}_{l-1}} = 0.$$

Since $n = (m_7-l)(l-1) + (m_8+l-1)l$, comparing the coefficients of $x_{l-1}^{m_7-l} x_l^{m_8+l-1}$ in Equation (3.3), we get that

$$(3.5) \quad \frac{m_7+m_8-1}{m_7-l} N_l - N_{l,\mathbf{e}_{l-1}} = 0.$$

Comparing Equations (3.4) and (3.5), we get that $N_l = 0 = N_{l, \mathbf{e}_{l-1}}$.

To show that $N_{l, \mathbf{s}} = 0$ for all $\mathbf{s} \in Q(l)$, order the set $Q(l)$ lexicographically and induct on \mathbf{s} . The same argument, as in Case 1, shows that $N_{l, \mathbf{s}} = 0$ for all $\mathbf{s} \in Q(l)$. Therefore P_l must contain a monomial from the set A_l . \square

Taking $c_1 = c_2 = 1$ in Proposition 3.1, we get

Corollary 3.6. *Every nonzero homogeneous polynomial of degree at most $2n$ in $\mathcal{I}_{n,l}$ contains a monomial, with nonzero coefficient, that is divisible by $x_{l-1}x_l$ if $n \geq 2l^2 + l - 2$.*

As a consequence we prove the following theorem. This theorem can be viewed as a general property of the cohomology algebra of a complex Grassmann manifold.

Theorem 3.7. *For fixed $1 \leq i \leq l$, let $P_i \in \mathcal{I}_{n,l}$ be a nonzero homogeneous polynomial of degree $2(n - l + i)$. Assume that $n \geq 2l^2 + l - 2$. Then, for $k = i - 1, \dots, l$, P_i contains a monomial $\mathbf{x}^{\mathbf{n}}$ (\mathbf{n} depending on k), with nonzero coefficient, that is divisible by x_k .*

Proof. We induct on l . For $l = 1, 2$ the result follows from Corollary 3.6. Assume that for any $m \geq 2(l-1)^2 + (l-1) - 2$, if $P_j \in \mathcal{I}_{m, l-1}$, $1 \leq j \leq l-1$, is a nonzero homogeneous polynomial of degree $2(m - (l-1) + j)$ then P_j contains a monomial $\mathbf{x}^{\mathbf{n}}$ (\mathbf{n} depending on k), with nonzero coefficient, that is divisible by x_k , where $k = j - 1, \dots, l-1$. Consider the graded algebra homomorphism $f : \mathbb{Q}[x_1, \dots, x_l] \rightarrow \mathbb{Q}[x_1, \dots, x_{l-1}]$ defined by $f(x_i) = x_i$ for $i = 1, \dots, l-1$ and $f(x_l) = 0$. This induces a graded algebra homomorphism

$$\bar{f} : \frac{\mathbb{Q}[x_1, \dots, x_l]}{(R_1, \dots, R_l)} = H^*(G_{n,l}; \mathbb{Q}) \longrightarrow H^*(G_{n-1, l-1}; \mathbb{Q}) = \frac{\mathbb{Q}[x_1, \dots, x_{l-1}]}{(R'_1, \dots, R'_{l-1})}.$$

(This $\bar{f} = \hat{j}^*$ as mentioned in Fact(2) concerning the cohomology algebra of Grassmann manifold.) By Corollary 3.6, P_l contains a monomial $\mathbf{x}^{\mathbf{n}}$, with nonzero coefficient, that is divisible by $x_{l-1}x_l$. Hence we may assume that $i < l$. Then $f(P_i) \in (R'_1, \dots, R'_{l-1})$ is a homogeneous polynomial of degree $2(n - 1 - (l-1) + i) \leq 2(n-1)$. We claim that $f(P_i) \neq 0$. Suppose that $f(P_i) = 0$. Since $0 \neq P_i \in \mathcal{I}_{n,l}$ is of degree $2(n - l + i)$, $P_i = Q_1R_1 + \dots + Q_iR_i$ for some homogeneous polynomials $Q_1, \dots, Q_i \in \mathbb{Q}[x_1, \dots, x_l]$ with at least one $Q_r \neq 0$. Since $\deg Q_j = 2(i-j) < 2l$ for all $1 \leq j \leq i$, $Q_1, \dots, Q_i \in \mathbb{Q}[x_1, \dots, x_{l-1}]$. Let $s = \max\{j : Q_j \neq 0\}$. Applying f on P_i , we get $0 = f(P_i) = f(Q_1)f(R_1) + \dots + f(Q_i)f(R_i) = Q_1R'_1 + \dots + Q_sR'_s$. Since $Q_s \neq 0$, $Q_s \notin (R'_1, \dots, R'_{s-1})$ (due to degree reason). This contradicts the fact that R'_1, \dots, R'_s form a regular sequence in $\mathbb{Q}[x_1, \dots, x_{l-1}]$. Hence $f(P_i) \neq 0$.

Since $n \geq 2l^2 + l - 2$, $n - 1 \geq 2(l-1)^2 + (l-1) - 2$. Hence by induction hypothesis for each $k = i - 1, \dots, l-1$, $f(P_i)$ contains a monomial $\mathbf{x}^{\mathbf{n}}$ (\mathbf{n} depending on k), with nonzero coefficient, that is divisible by x_k . Thus P_i contains a monomial $\mathbf{x}^{\mathbf{n}}$, with nonzero coefficient, that is divisible by x_k , for $k = i - 1, \dots, l-1$. Therefore using Corollary 3.6, the result follows. \square

Next we prove the main theorem of this section.

Theorem 3.8. *Let $1 \leq k < l$ and $\bar{\phi} : H^*(G_{n,k}; \mathbb{Q}) \rightarrow H^*(G_{n,l}; \mathbb{Q})$ be a graded algebra homomorphism. Then $\bar{\phi}$ is trivial if $n \geq 2l^2 + l - 2$.*

Proof. Let $I = \mathcal{I}_{n,k} = (R_1, \dots, R_k)$, $I' = \mathcal{I}_{n,l} = (R'_1, \dots, R'_l)$, $H^*(G_{n,k}; \mathbb{Q}) = \frac{\mathbb{Q}[x_1, \dots, x_k]}{I}$ and $H^*(G_{n,l}; \mathbb{Q}) = \frac{\mathbb{Q}[y_1, \dots, y_l]}{I'}$. Suppose $\bar{\phi}(x_r + I) = Q_r + I'$, $r = 1, \dots, k$, for some homogeneous polynomials $Q_r \in \mathbb{Q}[y_1, \dots, y_l]$ of degree $2r$. Let $\phi : \mathbb{Q}[x_1, \dots, x_k] \rightarrow \mathbb{Q}[y_1, \dots, y_l]$ be a graded algebra homomorphism defined by $\phi(x_r) = Q_r$, for $1 \leq r \leq k$. Let $P_i = \phi(R_i)$. Therefore $P_i \in I'$ is a homogeneous polynomial of degree $2(n - k + i)$. Since ϕ is a graded algebra homomorphism, $\max\{j : y_j \text{ occurs in } P_i\} = k$. Hence, by Corollary 3.6, $P_i = 0$. Thus we get a graded algebra homomorphism $\phi : \mathbb{Q}[x_1, \dots, x_k] \rightarrow \mathbb{Q}[y_1, \dots, y_l]$ such that $\phi(R_i) = 0$ for all $i = 1, \dots, k$. Hence by Lemma 2.2, ϕ is trivial and thus $\bar{\phi}$ is trivial. \square

Since $H^*(G_{m-1,l}; \mathbb{Q})$ is a quotient of $H^*(G_{m,l}; \mathbb{Q})$, we obtain the following straightforward application of Theorem 3.8. An analogous result was obtained in [4, Theorem 1.1] when $m - l > n - k$ and $m - l \geq 2k^2 - k - 1$.

Corollary 3.9. *Let $1 \leq k < l$, $n \leq m$ and $\bar{\phi} : H^*(G_{n,k}; \mathbb{Q}) \rightarrow H^*(G_{m,l}; \mathbb{Q})$ be a graded algebra homomorphism. Then $\bar{\phi}$ is trivial if $n \geq 2l^2 + l - 2$.*

We give an example to show that the conclusion of Corollary 3.6 need not be satisfied for smaller values of n , while Theorem 3.8 is satisfied.

Example 3.10. Let $n = 6, l = 3$. Then

$$x_1^6 - 3x_1^4x_2 + 3x_1^2x_2^2 - 2x_2^3 = (3x_1^2 - 2x_2)R_1 + 2x_1R_2 \in \mathcal{I}_{6,3}$$

is a homogeneous polynomial of degree 12 which does not contain x_3 .

Since $x_1^n \in \mathcal{I}_{n,1}$ and $x_1^{k(n-k)} \notin \mathcal{I}_{n,k}$ for $k \geq 2$, any graded algebra homomorphism from $H^*(G_{n,1}; \mathbb{Q})$ to $H^*(G_{n,k}; \mathbb{Q})$ is trivial if $k \geq 2$. In particular, any graded algebra homomorphism from $H^*(G_{6,1}; \mathbb{Q})$ to $H^*(G_{6,3}; \mathbb{Q})$ is trivial. Also, from direct calculations, it follows that any graded algebra homomorphism from $H^*(G_{6,2}; \mathbb{Q})$ to $H^*(G_{6,3}; \mathbb{Q})$ is trivial.

4. GRADED ALGEBRA HOMOMORPHISMS FROM $H^*(G_{n,k}; \mathbb{Q})$ TO $H^*(G_{n,l}; \mathbb{Q})$ FOR $k > l$

In this section we consider the possible graded algebra homomorphisms from $H^*(G_{n,k}; \mathbb{Q})$ to $H^*(G_{n,l}; \mathbb{Q})$ where $k > l$. Unlike the case $k < l$, Example 4.1 illustrates that there exist nontrivial graded algebra homomorphisms from $H^*(G_{n,k}; \mathbb{Q})$ to $H^*(G_{n,l}; \mathbb{Q})$ if $k > l$, $l = 1$ and k divides n . For $k > l$, in certain cases, we prove that the only graded algebra homomorphism from $H^*(G_{n,k}; \mathbb{Q})$ to $H^*(G_{n,l}; \mathbb{Q})$ is trivial (Theorems 4.11 and 4.20). However, the problem of determining the possible graded algebra homomorphisms from $H^*(G_{n,k}; \mathbb{Q})$ to $H^*(G_{n,l}; \mathbb{Q})$ remains open for arbitrary $k > l$.

Example 4.1. Let $k > 1$. For $n = qk$, consider the graded algebra homomorphism $\phi : \mathbb{Q}[x_1, \dots, x_k] \rightarrow \mathbb{Q}[y]$ defined as $\phi(x_i) = 0$ for $i < k$ and $\phi(x_k) = cy^k$ for some $0 \neq c \in \mathbb{Q}$. Suppose $\phi(R_i) \neq 0$ for some i , $1 \leq i \leq k$. Note that $\phi(R_i) \neq 0$ if and only if R_i contains a monomial of the form x_k^m , for some $1 \leq m \leq q$, with nonzero coefficient. Thus $\deg R_i = 2(n - k + i) = 2km$. Hence $i = k(m - q + 1) \leq 0$ unless $m = q$. Thus $i = k$. Also, $\phi(R_k) = (-1)^q c^q y^n$. Therefore ϕ induces a nontrivial map $\bar{\phi}$

$$\bar{\phi} : \frac{\mathbb{Q}[x_1, \dots, x_k]}{(R_1, \dots, R_k)} = H^*(G_{n,k}; \mathbb{Q}) \rightarrow H^*(G_{n,1}; \mathbb{Q}) = \frac{\mathbb{Q}[y]}{(y^n)}.$$

Now we provide sufficient conditions which will assure the triviality of a graded algebra homomorphism from $H^*(G_{n,k}; \mathbb{Q})$ to $H^*(G_{n,l}; \mathbb{Q})$ for $1 < l < k$. We set $H^*(G_{n,k}; \mathbb{Q}) = \frac{\mathbb{Q}[x_1, \dots, x_k]}{(R_1, \dots, R_k)}$, $H^*(G_{n,l}; \mathbb{Q}) = \frac{\mathbb{Q}[y_1, \dots, y_l]}{(R'_1, \dots, R'_l)}$, $I = \mathcal{I}_{n,k} = (R_1, \dots, R_k)$ and $I' = \mathcal{I}_{n,l} = (R'_1, \dots, R'_l)$.

Theorem 4.2. *Let $1 < l < k$ and $\phi : \mathbb{Q}[x_1, \dots, x_k] \rightarrow \mathbb{Q}[y_1, \dots, y_l]$ be a graded algebra homomorphism such that $\phi(\mathcal{I}_{n,k}) \subseteq \mathcal{I}_{n,l}$. Suppose $(\phi(x_1), \dots, \phi(x_k)) \subseteq (y_1, \dots, y_{l-2})$. Then ϕ is trivial if $n \geq 2l^2 + kl - 2$.*

Proof. By Lemma 2.2, it suffices to show that $\phi(R_i) = 0$ for all $i = 1, \dots, k$. Let $\phi(R_i) = P_i$ for some $P_i \in I'$. Since $\deg P_i < 2(n - l + 1)$ for $i = 1, \dots, k - l$, $P_i = 0$ for $i = 1, \dots, k - l$. Suppose $P_{k-l+i} \neq 0$ for some $i \geq 1$. Since $n - l + 1 \geq l(l - 1) + kl + (l + 1)(l - 1)$, by Proposition 3.1(2), P_{k-l+i} contains a monomial $y_{l-1}^{m_1} y_l^{m_2}$ or $\mathbf{y}^s y_{l-1}^{m_1} y_l^{m_2}$, where $m_1 \geq 1$, $m_2 \geq k$ and $\text{wt } \mathbf{s} < i \leq l$, with nonzero coefficient. Since $\phi(x_i)$ does not contain a monomial of the form $y_{l-1}^{m_1} y_l^{m_2}$ with nonzero coefficient, P_{k-l+i} contains a monomial $\mathbf{y}^s y_{l-1}^{m_1} y_l^{m_2}$ with $m_1 \geq 1$, $m_2 \geq k$ and $\text{wt } \mathbf{s} < l$. We have

$$(4.3) \quad P_{k-l+i} = \phi(R_{k-l+i}) = \sum_{\substack{\mathbf{n} \in \mathbb{N}^k, \text{wt } \mathbf{n} = n-l+i, \\ n_{l+1} + \dots + n_k < l}} \phi(c_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}) + \sum_{\substack{\mathbf{n} \in \mathbb{N}^k, \text{wt } \mathbf{n} = n-l+i, \\ n_{l+1} + \dots + n_k \geq l}} \phi(c_{\mathbf{n}} \mathbf{x}^{\mathbf{n}}).$$

If $n_{l+1} + \dots + n_k \geq l$ then for every monomial $\mathbf{y}^{s'} y_{l-1}^{m_3} y_l^{m_4}$ in $\phi(\mathbf{x}^n)$, $\text{wt } \mathbf{s}' \geq l$. Hence $\mathbf{y}^s y_{l-1}^{m_1} y_l^{m_2}$ does not occur in the second summand of (4.3). Suppose $n_{l+1} + \dots + n_k < l$. Since the highest possible power of y_l in a monomial in $\phi(x_j)$ is $\lfloor \frac{k}{l} \rfloor$ for all $j = 1, \dots, k$ and $\phi(x_l)$ does not contain y_l with nonzero coefficient, the power of y_l in a monomial in $\phi(\mathbf{x}^n)$ is $\leq (n_{l+1} + \dots + n_k) \lfloor \frac{k}{l} \rfloor < l \lfloor \frac{k}{l} \rfloor \leq k$. This implies that the power of y_l in a monomial in $\phi(\mathbf{x}^n) < k$. Hence $\mathbf{y}^s y_{l-1}^{m_1} y_l^{m_2}$ does not occur in the first summand of (4.3). This implies that $\phi(R_{k-l+i})$ does not contain $\mathbf{y}^s y_{l-1}^{m_1} y_l^{m_2}$, which leads to a contradiction. \square

Corollary 4.4. *Let $1 < l < k$ and $\bar{\phi} : H^*(G_{n,k}; \mathbb{Q}) \rightarrow H^*(G_{n,l}; \mathbb{Q})$ be a graded algebra homomorphism. For $i = 1, \dots, k$, let $\bar{\phi}(x_i + \mathcal{I}_{n,k}) = Q_i + \mathcal{I}_{n,l}$ for some homogeneous polynomials $Q_i \in \mathbb{Q}[y_1, \dots, y_l]$. Suppose $(Q_1, \dots, Q_k) \subseteq (y_1, \dots, y_{l-2})$. Then $\bar{\phi}$ is trivial if $n \geq 2l^2 + kl - 2$.*

Proof. Let $\phi : \mathbb{Q}[x_1, \dots, x_k] \rightarrow \mathbb{Q}[y_1, \dots, y_l]$ be a graded algebra homomorphism defined as $\phi(x_i) = Q_i$. Since ϕ induces a map $\bar{\phi}$, $\phi(I) \subseteq I'$. Therefore by Theorem 4.2, the result follows. \square

The following theorem generalizes Theorem 4.2.

Theorem 4.5. *Let $1 < l < k$ and $\phi : \mathbb{Q}[x_1, \dots, x_k] \rightarrow \mathbb{Q}[y_1, \dots, y_l]$ be a graded algebra homomorphism such that $\phi(\mathcal{I}_{n,k}) \subseteq \mathcal{I}_{n,l}$. Suppose $(\phi(x_1), \dots, \phi(x_k)) \subseteq (y_1, \dots, y_{l-1})$. Then ϕ is trivial if $n \geq 3k^2 - 2$.*

Proof. Let $\psi : \mathbb{Q}[y_1, \dots, y_l] \rightarrow \mathbb{Q}[y_{l-1}, y_l]$ be a graded algebra homomorphism defined as $\psi(y_i) = 0$ for $i = 1, \dots, l-2$, $\psi(y_{l-1}) = y_{l-1}$ and $\psi(y_l) = y_l$. Let $J = (\psi(R'_1), \dots, \psi(R'_l))$ and $h = \psi \circ \phi$. As $n \geq 3k^2 - 2$, so $n \geq 2l^2 + kl - 2$. Hence by Theorem 4.2, it suffices to show that h is trivial. We show that $h(R_i) = 0$ for all $i = 1, \dots, k$. Since $\deg R_i < 2(n - l + 1)$ for $i = 1, \dots, k - l$ and $\phi(R_i) \in I'$, $\phi(R_i) = 0$ for $1 \leq i \leq k - l$ and hence $h(R_i) = 0$ for $1 \leq i \leq k - l$. Let $\psi(R'_i) = Q_i$ for $i = 1, \dots, l$. Since $\phi(R_i) \in I'$, $h(R_i) \in J$ for all $1 \leq i \leq k$. Therefore

$$(4.6) \quad h(R_{k-l+i}) = \lambda_i Q_i \text{ for } 1 \leq i < l \text{ and}$$

$$(4.7) \quad h(R_k) = \lambda_l Q_l + \lambda'_l y_{l-1} Q_1.$$

We claim that Q_i contains a monomial $y_{l-1}^r y_l^s$ with $1 \leq r \leq l$, $s \geq l$. Since $n - l + i \geq n - l + 1 \geq n - k + 1 \geq 3k^2 - k - 1 \geq 2k^2 - 1 \geq l(l-1) + l^2 + (l-1)$, by Proposition 2.3, there exist integers $r \geq 1, s \geq l$, such that $n - l + i = r(l-1) + sl$. Hence Q_i contains a monomial $y_{l-1}^r y_l^s$. If $r > l$, then replace r by $r_1 = r - l$ and s by $s_1 = s + l - 1$. If $r_1 \leq l$ then we are done. Otherwise continue as above. This proves the claim.

Note that if \mathbf{x}^n occurs in R_{k-l+i} , $1 \leq i \leq l$, then $|\mathbf{n}| \geq 2k$. Suppose $|\mathbf{n}| < 2k$. Then $\deg \mathbf{x}^n = 2n_1 + \dots + 2kn_k \leq 2k(n_1 + \dots + n_k) < 2k(2k)$. But $\deg R_{k-l+i} \geq 2(n - l + 1) \geq 2(3k^2 - k - 1) \geq 2(2k^2)$, a contradiction. Hence $|\mathbf{n}| \geq 2k$. Since $\phi(x_j)$ does not contain a pure monomial in y_l and $\psi(y_j) = 0$ for $1 \leq j < l-1$, $h(\mathbf{x}^n)$ is sum of monomials of the form $y_{l-1}^{m_1} y_l^{m_2}$ with $m_1 \geq 2k > k > l$, for a monomial \mathbf{x}^n occurring in R_{k-l+i} . Hence comparing the coefficients of $y_{l-1}^r y_l^s$, where $1 \leq r \leq l$, $s \geq l$ in Equation (4.6) we get that $\lambda_i = 0$.

Let $i = l$. If $h(\mathbf{x}^n)$ contains a monomial $y_{l-1}^r y_l^s$ with nonzero coefficient then $r \geq 2k > 2l$. Note that Q_l contains a monomial of the form $y_{l-1}^r y_l^s$, with $1 \leq r \leq l$ and $s \geq l$. Let $r_1 = r + l, s_1 = s - (l-1)$. Then comparing the coefficients of $y_{l-1}^r y_l^s$ and $y_{l-1}^{r_1} y_l^{s_1}$ in Equation (4.7), we get that

$$(4.8) \quad \frac{(r+s)!}{r!s!} \lambda_l - \frac{(r+s-1)!}{(r-1)!s!} \lambda'_l = 0$$

$$(4.9) \quad \frac{(r_1+s_1)!}{r_1!s_1!} \lambda_l - \frac{(r_1+s_1-1)!}{(r_1-1)!s_1!} \lambda'_l = 0.$$

Solving Equations (4.8) and (4.9), we get that $\lambda_l = \lambda'_l = 0$. Hence $h(R_k) = 0$. Therefore by Lemma 2.2, h is trivial. \square

Corollary 4.10. *Let $\bar{\phi} : H^*(G_{n,k}; \mathbb{Q}) \rightarrow H^*(G_{n,l}; \mathbb{Q})$, $1 < l < k$ be a graded algebra homomorphism. For $i = 1, \dots, k$, let $\bar{\phi}(x_i + \mathcal{I}_{n,k}) = Q_i + \mathcal{I}_{n,l}$ for some homogeneous polynomials $Q_i \in \mathbb{Q}[y_1, \dots, y_l]$. Suppose $(Q_1, \dots, Q_k) \subseteq (y_1, \dots, y_{l-1})$. Then $\bar{\phi}$ is trivial if $n \geq 3k^2 - 2$.*

Proof. Using argument as in Corollary 4.4, the result follows from Theorem 4.5. \square

As a consequence we determine the possible graded algebra homomorphisms from $H^*(G_{n,k}; \mathbb{Q})$ to $H^*(G_{n,l}; \mathbb{Q})$ when $2 < l < k < 2(l-1)$.

Theorem 4.11. *Let $2 < l < k < 2(l-1)$ and $\bar{\phi} : H^*(G_{n,k}; \mathbb{Q}) \rightarrow H^*(G_{n,l}; \mathbb{Q})$ be a graded algebra homomorphism. Then $\bar{\phi}$ is trivial if $n \geq 3k^2 - 2$.*

Proof. For $i = 1, \dots, k$, let $\bar{\phi}(x_i + I) = P_i + I'$ with $\deg P_i = \deg x_i$. Let $\phi : \mathbb{Q}[x_1, \dots, x_k] \rightarrow \mathbb{Q}[y_1, \dots, y_l]$ be a graded algebra homomorphism defined as $\phi(x_i) = P_i$. By Theorem 4.5, it suffices to show that $(P_1, \dots, P_k) \subseteq (y_1, \dots, y_{l-1})$. Let $\psi : \mathbb{Q}[y_1, \dots, y_l] \rightarrow \mathbb{Q}[y_{l-1}, y_l]$ be a graded algebra homomorphism defined as $\psi(y_i) = 0$ for $i = 1, \dots, l-2$ and $\psi(y_i) = y_i$ for $i = l-1, l$. Let $h = \psi \circ \phi$. Since $k < 2(l-1)$, P_i does not contain monomials of the form $y_{l-1}^{m_1} y_l^{m_2}$ with $m_1 + m_2 > 1$. Hence $h(x_i) = 0$ for $i = 1, \dots, l-2, l+1, \dots, k$. Let $h(x_{l-1}) = ay_{l-1}$ and $h(x_l) = by_l$ for some $a, b \in \mathbb{Q}$. Since $\deg R_i < 2(n-l+1)$ for $i = 1, \dots, k-l$ and $h(R_i) = \psi(\phi(R_i)) \in (\psi(R'_1), \dots, \psi(R'_l))$, $h(R_i) = 0$ for $i = 1, \dots, k-l$. Since $n-k+1 \geq k(k-1) + (k-1) + k \geq l(l-1) + (l-1) + l$, by Proposition 2.3, there exist integers $r_1 \geq 1$ and $s_1 \geq 1$ such that $n-k+1 = r_1(l-1) + s_1l$. Hence comparing the coefficients of $y_{l-1}^{r_1} y_l^{s_1}$ in $h(R_1)$, we get $a^{r_1} b^{s_1} = 0$. Suppose $b \neq 0$. Then $a = 0$. This implies

$$h(R_i) = \begin{cases} c_i y_l^{m_i} & \text{for some } c_i \in \mathbb{Q} \text{ and } m_i \in \mathbb{N} \\ 0 & \text{otherwise.} \end{cases} \quad \begin{array}{l} \text{if } l \text{ divides } (n-k+i) \\ \text{otherwise.} \end{array}$$

Let $Q_i = \psi(R'_i)$. Thus

$$(4.12) \quad h(R_{k-l+i}) = \lambda_i Q_i \text{ for } 1 \leq i < l \text{ and}$$

$$(4.13) \quad h(R_k) = \lambda_l Q_l + \lambda'_l y_{l-1} Q_1.$$

Since $n-l+i \geq l(l-1) + (l-1) + l$, by Proposition 2.3, there exist integers $r_2 \geq 1$ and $s_2 \geq 1$ such that $n-l+i = r_2(l-1) + s_2l$. Hence comparing the coefficients of $y_{l-1}^{r_2} y_l^{s_2}$ in Equation (4.12), we get $\lambda_i = 0$ for $1 \leq i < l$. Thus $h(R_{k-l+i}) = 0$ for $1 \leq i < l$. Let $h(R_k) = c y_l^s$. Then $n = sl = l(l-1) + (s-l+1)l = 2l(l-1) + (s-2l+2)l$. Since $sl = n \geq 2l^2 - 1 \geq 2l^2 - 2l$, $s \geq 2l-2$. Thus $s-2l+2 \geq 0$ which implies that $s-l+1 \geq 0$. Hence comparing the coefficients of $y_{l-1}^l y_l^{s-l+1}$ and $y_{l-1}^{2l} y_l^{s-2l+2}$ in Equation (4.13), we get

$$(4.14) \quad \frac{(s+1)!}{l!(s-l+1)!} \lambda_l - \frac{s!}{(l-1)!(s-l+1)!} \lambda'_l = 0$$

$$(4.15) \quad \frac{(s+2)!}{(2l)!(s-2l+2)!} \lambda_l - \frac{(s+1)!}{(2l-1)!(s-2l+2)!} \lambda'_l = 0.$$

Solving Equation (4.14) and (4.15) we get $\lambda_l = \lambda'_l = 0$. Hence $h(R_k) = 0$. Note that $x_l^{s_j}$ occurs with nonzero coefficient in R_{k-l+j} for some $1 \leq j \leq l$, where s_j is a positive integer. Hence comparing the coefficients of $y_l^{s_j}$ in $h(R_{k-l+j})$, we get $b = 0$, a contradiction. Hence $b = 0$. \square

Now, as an application of Corollary 4.10, we shall attain another sufficient condition for the triviality of a graded algebra homomorphism from $H^*(G_{n,k}; \mathbb{Q})$ to $H^*(G_{n,l}; \mathbb{Q})$.

Discussion 4.16. Let $\rho : \mathbb{Q}[y_1, \dots, y_l] \rightarrow \mathbb{Q}[y_l]$ be a graded algebra homomorphism defined by $\rho(y_i) = 0$ for $i = 1, \dots, l-1$ and $\rho(y_l) = y_l$. Let $T_i = \rho(R'_i)$. Then ρ induces a graded algebra

homomorphism

$$(4.17) \quad \bar{\rho} : \frac{\mathbb{Q}[y_1, \dots, y_l]}{(R'_1, \dots, R'_l)} \rightarrow \frac{\mathbb{Q}[y_l]}{(T_1, \dots, T_l)}.$$

Let $\bar{\phi} : H^*(G_{n,k}; \mathbb{Q}) \rightarrow H^*(G_{n,l}; \mathbb{Q})$, $n \geq 3k^2 - 2$, be a graded algebra homomorphism defined by $\bar{\phi}(x_i + I) = P_i + I'$, where $\deg P_i = \deg x_i$. Therefore, we get the following graded algebra homomorphism

$$(4.18) \quad \bar{\rho} \circ \bar{\phi} : \frac{\mathbb{Q}[x_1, \dots, x_k]}{(R_1, \dots, R_k)} \rightarrow \frac{\mathbb{Q}[y_l]}{(T_1, \dots, T_l)}.$$

Therefore, in order to show that $\bar{\phi}$ is trivial, by Corollary 4.10, it suffices to show that $(P_1, \dots, P_k) \subset (y_1, \dots, y_{l-1})$ which is equivalent to the triviality of the map $\bar{\rho} \circ \bar{\phi}$.

Let $k = el + f$ and $n = e_1l + f_1$ for some integers e, e_1, f, f_1 , where $0 \leq f, f_1 < l$. Note that, if j is not a multiple of l then $\bar{\rho} \circ \bar{\phi}(x_j + \mathcal{I}_{n,k}) = 0$. Let $\bar{\rho} \circ \bar{\phi}(x_{il} + \mathcal{I}_{n,k}) = \tau_i y_l^i + (T_1, \dots, T_l)$ for $i = 1, \dots, e$. Also,

$$T_i = \begin{cases} (-1)^{e_1} y_l^{e_1} & \text{if } i = l - f_1 \\ 0 & \text{otherwise.} \end{cases}$$

Let R_{t_1}, \dots, R_{t_s} ($s = e$ or $e + 1$) be those R_i 's which have degrees that are multiples of $2l$. Let $\deg R_{t_j} = 2lq_j$, for $j = 1, \dots, s$, for some integers q_j . Then for $j = 1, \dots, s$,

$$q_j = \begin{cases} e_1 - e + j & \text{if } s = e \\ e_1 - e + (j - 1) & \text{if } s = e + 1. \end{cases}$$

Let $S_j = \bar{\rho} \circ \bar{\phi}(R_{t_j})$. Then

$$S_j = \left(\sum_{\mathbf{n} \in \mathbb{N}^e, \text{wt } \mathbf{n} = q_j} (-1)^{|\mathbf{n}|} c_{\mathbf{n}} \tau^{\mathbf{n}} \right) y_l^{q_j} + (T_1, \dots, T_l).$$

Let $S'_j := \sum_{\mathbf{n} \in \mathbb{N}^e, \text{wt } \mathbf{n} = q_j} (-1)^{|\mathbf{n}|} c_{\mathbf{n}} \tau^{\mathbf{n}}$. Since $S_j = 0$ for $j \neq s$, we get

$$(4.19) \quad S'_j = 0 \text{ for } j \neq s.$$

Define a graded algebra homomorphism $\theta : \mathbb{Q}[x_1, \dots, x_e] \rightarrow \mathbb{Q}[y]$ by $\theta(x_i) = \tau_i y^i$, where $\deg y = 2$. Let R'_j be the part of degree $2(e_1 - e + j)$ of the formal inverse of $c = 1 + x_1 + \dots + x_e$. Thus, $\mathcal{I}_{e_1, e} = (R'_1, \dots, R'_e)$. Now, if $s = e$, then

$$\theta(R'_j) = \theta \left(\sum_{\mathbf{n} \in \mathbb{N}^e, \text{wt } \mathbf{n} = q_j} (-1)^{|\mathbf{n}|} c_{\mathbf{n}} \mathbf{x}^{\mathbf{n}} \right) = \left(\sum_{\mathbf{n} \in \mathbb{N}^e, \text{wt } \mathbf{n} = q_j} (-1)^{|\mathbf{n}|} c_{\mathbf{n}} \tau^{\mathbf{n}} \right) y^{q_j} = S'_j y^{q_j} = 0$$

for $j = 1, \dots, e - 1$. Then θ induces a graded algebra homomorphism from $H^*(G_{e_1, e}; \mathbb{Q})$ to $H^*(G_{e_1, 1}; \mathbb{Q})$. Therefore, if any graded algebra homomorphism from $H^*(G_{e_1, e}; \mathbb{Q})$ to $H^*(G_{e_1, 1}; \mathbb{Q})$ is trivial, then the homomorphism in (4.18) is trivial. \square

For $s = e + 1$, we obtain the following theorem.

Theorem 4.20. *Let $1 < l < k$, $k = el + f$ and $n = e_1l + f_1$ for some integers e, f, e_1, f_1 such that $0 \leq f, f_1 < l$. Let $\bar{\phi} : H^*(G_{n,k}; \mathbb{Q}) \rightarrow H^*(G_{n,l}; \mathbb{Q})$ be a graded algebra homomorphism. Then $\bar{\phi}$ is trivial if $f > f_1$ and $n \geq 3k^2 - 2$.*

Proof. Let the notations be as in Discussion 4.16. Since $f > f_1$, $s = e + 1$. Hence, by Equation (4.19), $S'_j = 0$ for $j = 1, \dots, e$. Define a graded algebra homomorphism $\theta : \mathbb{Q}[x_1, \dots, x_e] \rightarrow \mathbb{Q}[y]$

by $\theta(x_i) = \tau_i y^i$, where $\deg y = 2$. Let $\mathcal{I}_{e_1-1,e} = (R_1'', \dots, R_e'')$, where R_j'' is the part of degree $2(e_1 - 1 - e + j)$ of the formal inverse of $c = 1 + x_1 + \dots + x_e$. Then

$$\theta(R_j'') = S_j' y^{e_1-1-e+j} = 0$$

for $j = 1, \dots, e$. Thus by Lemma 2.2, θ is trivial, which implies that $\tau_i = 0$ for all i . Hence $\bar{\rho} \circ \bar{\phi}$ is trivial. Therefore, by Corollary 4.10, $\bar{\phi}$ is trivial. \square

Remark 4.21. Note that if $f \leq f_1$, then the number of R_i 's having degrees multiple of $2l$ is e , i.e. $s = e$. Thus in order to show that $\bar{\phi}$ is trivial, from Discussion 4.16, it suffices to show that every graded algebra homomorphism from $H^*(G_{e_1,e}; \mathbb{Q})$ to $H^*(G_{e_1,1}; \mathbb{Q})$ is trivial. But, from the existence of a nontrivial graded algebra homomorphism from $H^*(G_{e_1,e}; \mathbb{Q})$ to $H^*(G_{e_1,1}; \mathbb{Q})$, we cannot conclude that $\bar{\phi}$ is trivial.

5. PROOF OF THEOREM 1.1

At first, we recall a relation between the homotopy class of a map and the homomorphism it induces in cohomology with rational coefficients. Familiarity with basic notions in rational homotopy theory has been assumed. For further details, see [5, 10].

Let X be any simply connected finite CW complex and let X_0 denote its rationalization. Denoting the minimal model of X by \mathcal{M}_X , one has a bijection $[X_0, Y_0] \cong [\mathcal{M}_Y, \mathcal{M}_X]$, $[h] \mapsto [\Phi_h]$, where on the left we have homotopy classes of continuous maps $X_0 \rightarrow Y_0$ and on the right we have homotopy classes of differential graded commutative algebra homomorphisms of the minimal models $\mathcal{M}_Y \rightarrow \mathcal{M}_X$. In the case when $X = U(n)/(U(n_1) \dots \times U(n_r))$ is a complex flag manifold (i.e. when $n = \sum_{i=1}^r n_i$), then X is a Kähler manifold and hence is formal, that is there is a morphism of differential graded commutative algebras $\rho_X : \mathcal{M}_X \rightarrow H^*(X; \mathbb{Q})$ which induces an isomorphism in cohomology, where $H^*(X; \mathbb{Q})$ is endowed with the zero differential. Moreover, it is known that when both X and Y are complex flag manifolds, any continuous map $h : X \rightarrow Y$ is formal, that is, the homotopy class of the morphism $h_0 : X_0 \rightarrow Y_0$ is determined by the graded \mathbb{Q} -algebra homomorphism $h^* : H^*(Y; \mathbb{Q}) \rightarrow H^*(X; \mathbb{Q})$. The above discussion can be written as the following theorem.

Theorem 5.1. ([9], Theorem 1.1) *Let X, Y be complex flag manifolds. Then $[h] \mapsto H^*(h; \mathbb{Q})$ establishes a bijection from $[X_0, Y_0]$ to the set of graded \mathbb{Q} -algebra homomorphisms $\text{Hom}_{\text{alg}}(H^*(Y; \mathbb{Q}), H^*(X; \mathbb{Q}))$.*

We now turn to the proof of Theorem 1.1.

Proof of Theorem 1.1. Let $h : G_{n,l} \rightarrow G_{n,k}$ be a continuous map. By Theorem 3.8 and Theorem 4.11, h^* is trivial in Cases (i), (ii)(a). In Case (ii)(b), h^* satisfies the hypothesis of Theorem 4.20 with $f > 0$, $f_1 = 0$. So, h^* is trivial in Case (ii)(b). Then by Theorem 5.1, h_0 is null-homotopic in all three cases. \square

Remark 5.2. (1) The above proof also establishes the following statement, which is a generalization of Case (ii)(b) of Theorem 1.1.

Any continuous map $\phi : G_{n,l} \rightarrow G_{n,k}$ is rationally null homotopic if $1 < l < k$, $f > f_1$, $n \geq 3k^2 - 2$, where $k = el + f$ and $n = e_1 l + f_1$ for some integers e, f, e_1, f_1 such that $0 \leq f, f_1 < l$.

(2) The set of homotopy classes of continuous maps $[G_{n,l}, G_{n,k}]$ is finite when n, k, l are as in Theorem 1.1 and Remark 5.2(1). This follows from the finiteness of the set of homotopy classes of continuous maps $h : X \rightarrow Y$ having the same rationalization $h_0 : X_0 \rightarrow Y_0$. See [20, §12].

As an application of Theorem 1.1, we get the following *invariant subspace theorem*. If $y \in G_{n,k}$ and $x \in G_{n,l}$ and $k < l$, then it makes sense to write $y \subset x$.

Theorem 5.3. *Let $h : G_{n,l} \rightarrow G_{n,k}$ be any continuous map where n, k, l are as in Theorem 1.1(i). Then there exists an element $x \in G_{n,l}$ such that $h(x) \subset x$.*

Proof. The proof of this theorem is analogous to that of [4, Theorem 1.3] with obvious changes. \square

6. EXAMPLES

In this section we provide a few examples, which do not follow from Theorems 3.8, 4.11, or 4.20, where the graded algebra homomorphisms between cohomology algebras of distinct Grassmann manifolds are trivial. We also give examples of nontrivial graded algebra homomorphisms from $H^*(G_{n,2}; \mathbb{Q})$ to $H^*(G_{n,1}; \mathbb{Q})$ even if n is not even (see Example 4.1 for n even).

In the following proposition we consider possible graded algebra homomorphisms from $H^*(G_{2m,3}; \mathbb{Q})$ to $H^*(G_{2m,2}; \mathbb{Q})$. Note that using Theorem 4.20, we can conclude that any such homomorphism is trivial provided $m \geq 13$. By direct calculations, we show that any such homomorphism is trivial for $m \geq 3$.

Proposition 6.1. *Let $\bar{\phi} : H^*(G_{2m,3}; \mathbb{Q}) \rightarrow H^*(G_{2m,2}; \mathbb{Q})$ be a graded algebra homomorphism. Then $\bar{\phi}$ is trivial if $m \geq 3$.*

Proof. In Example 6.4 (1), we shall show that any graded algebra homomorphism from $H^*(G_{6,3}; \mathbb{Q})$ to $H^*(G_{6,2}; \mathbb{Q})$ is trivial. Hence we assume that $m \geq 4$.

Let $I = \mathcal{I}_{2m,3} = (R_1, R_2, R_3)$, $I' = \mathcal{I}_{2m,2} = (R'_1, R'_2)$, $H^*(G_{2m,3}; \mathbb{Q}) = \frac{\mathbb{Q}[x_1, x_2, x_3]}{I}$ and $H^*(G_{2m,2}; \mathbb{Q}) = \frac{\mathbb{Q}[y_1, y_2]}{I'}$. Let $\bar{\phi}(x_1 + I) = a_{10}y_1 + I'$, $\bar{\phi}(x_2 + I) = a_{20}y_1^2 + a_{01}y_2 + I'$ and $\bar{\phi}(x_3 + I) = a_{30}y_1^3 + a_{11}y_1y_2 + I'$. Define $\phi : \mathbb{Q}[x_1, x_2, x_3] \rightarrow \mathbb{Q}[y_1, y_2]$ as $\phi(x_1) = a_{10}y_1$, $\phi(x_2) = a_{20}y_1^2 + a_{01}y_2$ and $\phi(x_3) = a_{30}y_1^3 + a_{11}y_1y_2$. It suffices to show that ϕ is trivial. Since $\phi(I) \subseteq I'$ and $\deg R_1 = 4m - 4$, $\phi(R_1) = 0$. Hence comparing the coefficients of y_2^{m-1} in $\phi(R_1)$, we get $a_{01} = 0$.

Case 1: $2m - 2 = 3q$ for some $q \in \mathbb{N}$.

In this case, x_3^q occurs in R_1 with nonzero coefficient. Hence comparing the coefficients of $(y_1y_2)^q$ in $\phi(R_1)$, we get $a_{11} = 0$. Thus $(\phi(x_1), \phi(x_2), \phi(x_3)) \subseteq (y_1)$. Next we shall prove that $\phi(R_2) = \phi(R_3) = 0$. Suppose $\phi(R_2) \neq 0$. Since $\phi(R_2) \in I'$, by Corollary 3.6, $\phi(R_2)$ contains a monomial $y_1^{m_1}y_2^{m_2}$ with nonzero coefficient and $m_2 > 0$, which is a contradiction. Hence $\phi(R_2) = 0$. Similarly, $\phi(R_3) = 0$. Hence by Lemma 2.2, ϕ is trivial.

Case 2: $2m - 2 = 3q + 1$ for some $q \in \mathbb{N}$.

In this case, $x_1x_3^q$ occurs in R_1 with nonzero coefficient. Hence comparing the coefficients of $y_1^{q+1}y_2^q$, we get $a_{10}a_{11}^q = 0$. If $a_{11} = 0$ then argument as in Case 1 gives the result. Suppose $a_{11} \neq 0$. Then $a_{10} = 0$. Now, comparing the coefficients of $y_1^{q+3}y_2^{q-1}$ in $\phi(R_1)$, we get $a_{11}^{q-1}a_{20}^2 = 0$. Hence $a_{20} = 0$. Thus $\phi(x_1) = \phi(x_2) = 0$. Since $\deg R_2$ is not a multiple of 6, $\phi(R_2) = 0$. Let

$$(6.2) \quad \phi(R_3) = \lambda y_1 R'_1 + \lambda' R'_2.$$

Since $\phi(R_3) = \phi((-1)^{q+1}x_3^{q+1}) = (-1)^{q+1}(a_{30}y_1^3 + a_{11}y_1y_2)^{q+1}$, comparing the coefficients of y_2^m , in Equation (6.2), we get $\lambda' = 0$. Then comparing the coefficients of $y_1^2y_2^{m-1}$, in Equation (6.2), we get $\lambda = 0$. Thus $\phi(R_3) = 0$. Hence by Lemma 2.2, ϕ is trivial.

Case 3: $2m - 2 = 3q + 2$ for some $q \in \mathbb{N}$.

Let

$$(6.3) \quad \phi(R_2) = \lambda R'_1.$$

Then comparing the coefficients of $y_1y_2^{m-1}$ in Equation (6.3), we get $\lambda = 0$. Hence $\phi(R_2) = 0$. Therefore comparing the coefficients of $(y_1y_2)^{q+1}$ in $\phi(R_2)$, we get $a_{11} = 0$. Then argument as in Case 1 completes the proof. \square

In Examples 6.4 (1) and (2), we show that all graded algebra homomorphisms from $H^*(G_{n,3}; \mathbb{Q})$ to $H^*(G_{n,2}; \mathbb{Q})$ are trivial, if n is 6 and 7 respectively. In Example 6.4 (3), we illustrate that the existence of a non-trivial graded algebra homomorphism between the cohomology algebras of distinct complex Grassmann manifolds depends upon the coefficient field.

Example 6.4. (1) Let $\bar{\phi} : H^*(G_{6,3}; \mathbb{Q}) \rightarrow H^*(G_{6,2}; \mathbb{Q})$ be a graded algebra homomorphism. Then $\bar{\phi}$ is trivial.

Proof: Let $I = \mathcal{I}_{6,3}, I' = \mathcal{I}_{6,2}$. Let $\bar{\phi}(x_1 + I) = a_{10}y_1 + I', \bar{\phi}(x_2 + I) = a_{20}y_1^2 + a_{01}y_2 + I'$ and $\bar{\phi}(x_3 + I) = a_{30}y_1^3 + a_{11}y_1y_2 + I'$. Define $\phi : \mathbb{Q}[x_1, x_2, x_3] \rightarrow \mathbb{Q}[y_1, y_2]$ by $\phi(x_1) = a_{10}y_1, \phi(x_2) = a_{20}y_1^2 + a_{01}y_2$ and $\phi(x_3) = a_{30}y_1^3 + a_{11}y_1y_2$. It suffices to show that ϕ is trivial. Since $\phi(R_1) = \phi(x_1^4 - 3x_1^2x_2 + 2x_1x_3 + x_2^2) = 0$, comparing the coefficients of y_2^2 , we get $a_{01} = 0$. Hence comparing the coefficients of $y_1^2y_2$ in $\phi(R_1)$, we get $a_{10}a_{11} = 0$. Suppose $a_{11} = 0$. Let $\phi(R_2) = \lambda R'_1$. Then comparing the coefficients of $y_1y_2^2$, we get $\lambda = 0$. Thus $\phi(R_2) = 0$. Let $\phi(R_3) = \lambda_1y_1R'_1 + \lambda_2R'_2$. Hence comparing the coefficients of y_2^3 , we get $\lambda_2 = 0$. Then comparing the coefficients of $y_1^2y_2^2$, we get $\lambda_1 = 0$. Thus $\phi(R_3) = 0$. Therefore by Lemma 2.2, ϕ is trivial.

Suppose $a_{11} \neq 0$. Then $a_{10} = 0$. Let $\phi(R_2) = \lambda R'_1$. Hence comparing the coefficients of $y_1y_2^2$, we get $\lambda = 0$. Thus $\phi(R_2) = 0$. Since $\phi(x_1) = a_{10}y_1 = 0$, $\phi(R_2) = \phi(2x_2x_3)$. This implies that $\phi(x_2)\phi(x_3) = 0$. Comparing the coefficients of $y_1^3y_2$, we get $a_{20} = 0$. Thus $\phi(x_2) = 0$. Hence $\phi(R_3) = \phi(x_3^2)$. Let $\phi(R_3) = \lambda_1y_1R'_1 + \lambda_2R'_2$. Then comparing the coefficients of y_2^3 , we get $\lambda_2 = 0$. Therefore $\phi(R_3) = \phi(x_3^2) = \lambda_1y_1R'_1$. Thus $(a_{30}y_1^3 + a_{11}y_1y_2)^2 = \lambda_1y_1(-y_1^5 + 4y_1^3y_2 - 3y_1y_2^2)$. Comparing the coefficients and solving, we get $\lambda_1 = 0$. Thus $\phi(R_3) = 0$. Therefore by Lemma 2.2, ϕ is trivial.

(2) Let $\bar{\phi} : H^*(G_{7,3}; \mathbb{Q}) \rightarrow H^*(G_{7,2}; \mathbb{Q})$ be a graded algebra homomorphism. Then $\bar{\phi}$ is trivial.

Proof: Let $I = \mathcal{I}_{7,3}, I' = \mathcal{I}_{7,2}$. Let $\bar{\phi}(x_1 + I) = ay_1 + I', \bar{\phi}(x_2 + I) = by_1^2 + cy_2 + I'$ and $\bar{\phi}(x_3 + I) = dy_1^3 + ey_1y_2 + I'$. Define $\phi : \mathbb{Q}[x_1, x_2, x_3] \rightarrow \mathbb{Q}[y_1, y_2]$ by $\phi(x_1) = ay_1, \phi(x_2) = by_1^2 + cy_2$ and $\phi(x_3) = dy_1^3 + ey_1y_2$. It suffices to show that ϕ is trivial. Since $\phi(R_1) = 0$, comparing the coefficients of $y_1y_2^2$, we get $c(2e - 3ac) = 0$. We claim that $c = 0$. Suppose $c \neq 0$. Then $2e = 3ac$. Let $\phi(R_2) = \lambda R'_1$. Comparing the coefficients of y_2^3 and $y_1^2y_2^2$, we get $c^3 = \lambda$ and $6a^2c^2 - 3ace - 3bc^2 + e^2 = 6\lambda = 6c^3$. Using $2e = 3ac$, we get $a^2 = \frac{4}{5}(b + 2c)$. Comparing the coefficients of $y_1^4y_2$ in $\phi(R_2)$,

$$-5a^4c + 4a^3e + 12a^2bc - 3abe - 3acd - 3b^2c + 2de = -5\lambda = -5c^3.$$

Using $2e = 3ac$ and $a^2 = \frac{4}{5}(b + 2c)$, we get

$$\frac{91}{25}b^2 + \left(\frac{64}{25} + 12\right)bc + \left(\frac{64}{25} + 5\right)c^2 = 0.$$

Thus $91\frac{b^2}{c^2} + 364\frac{b}{c} + 189 = 0$. Since $91x^2 + 364x + 189$ is irreducible over \mathbb{Q} , the equation $91x^2 + 364x + 189 = 0$ has no roots in \mathbb{Q} , a contradiction. Hence $c = 0$.

Let $\phi(R_2) = \lambda R'_1$. Then comparing the coefficients of y_2^3 , we get $\lambda = 0$. Thus $\phi(R_2) = 0$. Hence comparing the coefficients of $y_1^2y_2^2$, we get $e = 0$. Let $\phi(R_3) = \lambda_1y_1R'_1 + \lambda_2R'_2$. Then comparing the coefficients of $y_1y_2^2$ and $y_1^3y_2^2$, we get

$$(6.5) \quad -\lambda_1 + 4\lambda_2 = 0$$

$$(6.6) \quad 6\lambda_1 - 10\lambda_2 = 0.$$

Solving equations (6.5) and (6.6) gives $\lambda_1 = \lambda_2 = 0$. Thus $\phi(R_3) = 0$. Therefore by Lemma 2.2, ϕ is trivial.

(3) Let $\bar{\phi} : H^*(G_{5,2}; \mathbb{Q}) \rightarrow H^*(G_{5,1}; \mathbb{Q})$ be a graded algebra homomorphism. Then $\bar{\phi}$ is trivial. But there exists a nontrivial map $\bar{h} : H^*(G_{5,2}; \mathbb{R}) \rightarrow H^*(G_{5,1}; \mathbb{R})$.

Proof: Let $I = \mathcal{I}_{5,2}, I' = \mathcal{I}_{5,1}$. Let $\bar{\phi}(x_1 + I) = ay_1 + I'$ and $\bar{\phi}(x_2 + I) = by_1^2 + I'$. Define $\phi : \mathbb{Q}[x_1, x_2] \rightarrow \mathbb{Q}[y_1]$ by $\phi(x_1) = ay_1$ and $\phi(x_2) = by_1^2$. Then $\phi(I) \subset I'$. Since $\deg R_1 = 8 < 10$ and $\phi(R_1) \in (y_1^5)$, we get $\phi(R_1) = \phi(x_1^4 - 3x_1^2x_2 + x_2^2) = 0$. Thus $(a^4 - 3a^2b + b^2)y_1^4 = 0$ which implies that $a^4 - 3a^2b + b^2 = 0$. If $b = 0$ then $a = 0$ and hence $\bar{\phi}$ is trivial. Suppose $b \neq 0$. Then $\left(\frac{a^2}{b}\right)^2 - 3\frac{a^2}{b} + 1 = 0$.

Thus $\frac{a^2}{b} = \frac{3 \pm \sqrt{5}}{2} \notin \mathbb{Q}$. This implies that $\bar{\phi}$ is trivial. Define $h : \mathbb{R}[x_1, x_2] \rightarrow \mathbb{R}[y_1]$ by $h(x_1) = ay_1$ and $h(x_2) = \frac{2a^2}{3 \pm \sqrt{5}}y_1^2$. Then h induces a graded algebra homomorphism $\bar{h} : H^*(G_{5,2}; \mathbb{R}) \rightarrow H^*(G_{5,1}; \mathbb{R})$.

In section 4, we gave examples of nontrivial graded algebra homomorphisms $\bar{\phi} : H^*(G_{n,k}; \mathbb{Q}) \rightarrow H^*(G_{n,1}; \mathbb{Q})$ if k divides n (Example 4.1). In the next example we show that there do exist infinitely

many nontrivial graded algebra homomorphisms $\bar{\phi} : H^*(G_{n,2}; \mathbb{Q}) \rightarrow H^*(G_{n,1}; \mathbb{Q})$ even if n is not even. We need the following proposition for this purpose.

Proposition 6.7. *Let $n > 1$ be a positive integer. Consider the polynomial*

$$g(c_1, c_2) := \sum_{r+2s=n-1} (-1)^{r+s} \binom{r+s}{s} c_1^r c_2^s \in \mathbb{Z}[c_1, c_2],$$

where c_1 and c_2 are indeterminates. Then,

$$\begin{aligned} & \{(u, v) \in \mathbb{R}^2 - \{(0, 0)\} : g(u, v) = 0\} \\ = & \begin{cases} \{(\pm 2\sqrt{v} \cos(r\pi/n), v) : v > 0, r \in \mathbb{Z} \text{ and } e^{2ir\pi/n} \neq \pm 1\} \cup \{(0, v) : v \neq 0\} & \text{if } 2|n \\ \{(\pm 2\sqrt{v} \cos(r\pi/n), v) : v > 0, r \in \mathbb{Z} \text{ and } e^{2ir\pi/n} \neq \pm 1\} & \text{otherwise.} \end{cases} \end{aligned}$$

Let $A = \{(0, u) : u \in \mathbb{Q} - \{0\}\}$, $B = \{(\pm 2u, 2u^2) : u \in \mathbb{Q} - \{0\}\}$, $C = \{(\pm 3u, 3u^2) : u \in \mathbb{Q} - \{0\}\}$, $D = \{(\pm u, u^2) : u \in \mathbb{Q} - \{0\}\}$. Suppose $(0, 0) \neq (u, v) \in \mathbb{Q}^2$. Then $g(u, v) = 0$ if and only if

$$(6.8) \quad (u, v) \in \begin{cases} A & \text{if } n \equiv 2, 10 \pmod{12} \\ D & \text{if } n \equiv 3, 9 \pmod{12} \\ A \cup B & \text{if } n \equiv 4, 8 \pmod{12} \\ A \cup C \cup D & \text{if } n \equiv 6 \pmod{12} \\ A \cup B \cup C \cup D & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Further, if n is not a multiple of 2 or 3, then there are no non-zero rational solutions of $g(c_1, c_1)$.

Proof. By [21], we have

$$\sum_{s \geq 0} (-1)^s \binom{n-1-s}{s} (c_1 + c_2)^{n-1-2s} (c_1 c_2)^s = c_1^{n-1} + c_1^{n-2} c_2 + \cdots + c_2^{n-1}$$

in $\mathbb{Z}[c_1, c_2]$. Let $f(c_1, c_2) = \sum_{s \geq 0} (-1)^s \binom{n-1-s}{s} (c_1 + c_2)^{n-1-2s} (c_1 c_2)^s$. Then for $u, v, x, y \in \mathbb{C}$, we have

$$g(u, v) = \sum_{s \geq 0} (-1)^{n-1-s} \binom{n-1-s}{s} u^{n-1-2s} v^s = (-1)^{n-1} f(x, y),$$

where $x + y = u, xy = v$. Note that $(u, v) \neq (0, 0)$ if and only if $(x, y) \neq (0, 0)$. Also, $g(u, v) = 0$ if and only if $f(x, y) = 0$. Further, $(x, y) \neq (0, 0)$ and $f(x, y) = x^{n-1} + x^{n-2}y + \cdots + y^{n-1} = 0$ if and only if $x^n = y^n$ and $x \neq y$.

Let $v > 0$ and $e^{2ir\pi/n} \neq \pm 1$. Let $u = \pm 2\sqrt{v} \cos(r\pi/n)$. Since $x + y = u, xy = v$, we have $x, y = \frac{u \pm \sqrt{u^2 - 4v}}{2}$. Thus $(x, y) = (\sqrt{v}(e^{ir\pi/n}, e^{-ir\pi/n}), \sqrt{v}(e^{-ir\pi/n}, e^{ir\pi/n}), -\sqrt{v}(e^{ir\pi/n}, e^{-ir\pi/n})$ and $-\sqrt{v}(e^{-ir\pi/n}, e^{ir\pi/n})$, which implies that $x^n = y^n$ and $x \neq y$. Therefore $f(x, y) = 0$ and hence $g(u, v) = 0$. Moreover, if $2|n$, then each monomial in $g(c_1, c_2)$ contains c_1 , so $g(0, c_2) = 0$.

Let $g(u, v) = 0$ for some $(0, 0) \neq (u, v) \in \mathbb{R}^2$. Then $f(x, y) = 0$. Therefore $x = ye^{2ir\pi/n}$, where $e^{2ir\pi/n} \neq 1$. Hence, $u = x + y = y(1 + e^{2ir\pi/n}), v = xy = y^2 e^{2ir\pi/n} \in \mathbb{R}$ with $e^{2ir\pi/n} \neq 1$.

Thus

$$\frac{u^2}{v} = \frac{1 + 2e^{2ir\pi/n} + e^{4ir\pi/n}}{e^{2ir\pi/n}} = e^{-2ir\pi/n} + 2 + e^{2ir\pi/n} = 4\cos^2(r\pi/n).$$

This implies that

- 1) if $\cos(r\pi/n) \neq 0$, then $v > 0$ and $u = \pm 2\sqrt{v} \cos(r\pi/n)$ with $e^{2ir\pi/n} \neq \pm 1$ and
- 2) if $\cos(r\pi/n) = 0$, then $2|n$ and $u = 0$.

Let $(0, 0) \neq (u, v) \in \mathbb{Q}^2$ be such that $g(u, v) = 0$. Then $u = \pm 2\sqrt{v} \cos(r\pi/n)$ or 0. Let $\cos(r\pi/n) = \cos(m\pi/d)$ where $d|n$ and $(m, d) = 1$. Since $\sqrt{v} \cos(m\pi/d) \in \mathbb{Q}$ with $0 \neq v \in \mathbb{Q}$, we have $\cos^2(m\pi/d) \in \mathbb{Q}$ which means that $\cos(2m\pi/d) \in \mathbb{Q}$. Thus the only possible values for

$\cos(2m\pi/d)$ are $0, 1, -1, \frac{1}{2}, -\frac{1}{2}$ (see [15]). These correspond to $\cos(m\pi/d) = \pm 1/\sqrt{2}, \pm 1, 0, \pm \frac{\sqrt{3}}{2}, \pm \frac{1}{2}$ which in turn correspond to $\frac{m}{d} = \frac{1}{4}, \frac{3}{4}, 0, 1, \frac{1}{2}, \frac{1}{6}, \frac{5}{6}, \frac{1}{3}$ and $\frac{2}{3}$, respectively. We have omitted the case when $e^{2ir\pi/n} = 1$, i.e. when $\cos(m\pi/d) = 1, -1$ and $\frac{m}{d} = 0, 1$. If $\frac{m}{d} = \frac{1}{2}$, then $2|n$ and $\cos(m\pi/d) = 0$. Thus there are rational solutions as required whenever n is a multiple of 2 or 4 or 6 or 3. \square

Example 6.9. For fixed $(u, v) \in \mathbb{Q}^2 - \{(0, 0)\}$ as in (6.8), define $\phi : \mathbb{Q}[x_1, x_2] \rightarrow \mathbb{Q}[y]$ as $\phi(x_1) = uy$ and $\phi(x_2) = vy^2$. Then ϕ induces a nontrivial graded algebra homomorphism $\bar{\phi} : H^*(G_{n,2}; \mathbb{Q}) \rightarrow H^*(G_{n,1}; \mathbb{Q})$ and this gives the complete set of non-trivial graded algebra homomorphisms. Further, there are no non-trivial maps $\bar{\phi} : H^*(G_{n,2}; \mathbb{Q}) \rightarrow H^*(G_{n,1}; \mathbb{Q})$ if n is not divisible by 2 or 3.

Proof: We have $R_1 = \sum_{r+2s=n-1} (-1)^{r+s} \binom{r+s}{s} x_1^r x_2^s$. By Proposition 6.7, $\phi(R_1) = 0$ and thus ϕ gives a nontrivial graded algebra homomorphism $\bar{\phi} : H^*(G_{n,2}; \mathbb{Q}) \rightarrow H^*(G_{n,1}; \mathbb{Q})$. Also, these are the only possible graded algebra homomorphisms.

Suppose n is divisible by neither 2 nor 3. Let $\phi(x_1) = u_1y, \phi(x_2) = v_1y^2$ for some $u_1, v_1 \in \mathbb{Q}$. By Proposition 6.7, $g(u_1, v_1) \neq 0$. Hence

$$\phi(R_1) = \left(\sum_{r+2s=n-1} (-1)^{r+s} \binom{r+s}{s} c_1^r c_2^s \right) y^{n-1} = g(u_1, v_1) y^{n-1} \neq 0.$$

Therefore there are no non-trivial graded algebra homomorphisms from $H^*(G_{n,2}; \mathbb{Q})$ to $H^*(G_{n,1}; \mathbb{Q})$.

ACKNOWLEDGEMENTS

We thank Prof. P. Sankaran for suggesting the problem and many useful discussions. We thank Prof. B. Sury for giving proofs of Proposition 6.7 and Example 6.9. We also thank the referees for a careful reading of the manuscript and giving suggestions which have improved our manuscript.

REFERENCES

- [1] Borel, A.: *Sur la cohomologie des espaces fibrés principaux et des espaces homogènes de groupes de Lie compacts*, Ann. of Math. (2) **57** (1953), 115–207.
- [2] Bott, R.—Tu, L.: *Differential forms in algebraic topology*, GTM **82**, Springer-Verlag, New York, 1982.
- [3] Brewster, S.—Homer, W.: *Rational automorphisms of Grassmann manifolds*, Proc. Amer. Math. Soc., **88** (1983), 181–183.
- [4] Chakraborty, P.—Sankaran, P.: *Maps between certain complex Grassmann manifolds*, Topology and its Applications, **170** (2014), 119–123.
- [5] Félix, Y.—Halperin, S.—Thomas, J. C.: *Rational homotopy theory*, GTM **205**, Springer-Verlag, New York, 2001.
- [6] Friedlander, E.: *Maps between localized homogeneous spaces*, Topology, **16** (1977), 205–216.
- [7] Fulton, W.: *Intersection theory*, second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete. **3**, Springer-Verlag, Berlin, 1998.
- [8] Glover, H.—Homer, W.: *Endomorphisms of the cohomology ring of finite Grassmann manifolds*, In: Geometric applications of homotopy theory, Lecture Notes in Math., **657**, Springer, Berlin, 1978, pp. 170–193.
- [9] Glover, H. H.—Homer, W.: *Self-maps of flag manifolds*, Trans. Amer. Math. Soc., **267** (1981), 423–434.
- [10] Griffiths, P.—Morgan, J.: *Rational homotopy theory and differential forms*, Progress in Mathematics **16**, Birkhäuser, Boston, 1981.
- [11] Hoffman, M.: *Endomorphisms of the cohomology of complex Grassmannians*, Trans. Amer. Math. Soc., **281** (1984), 745–760.
- [12] Hoffman, M.—Homer, W.: *On cohomology automorphisms of complex flag manifolds*, Proc. Amer. Math. Soc., **91** (1984), 643–648.
- [13] Korbaš, J.—Sankaran, P.: *On continuous maps between Grassmann manifolds*, Proc. Ind. Acad. Sci. Math. Sci., **101** (1991), 111–120.
- [14] Milnor, J.—Stasheff, J.: *Characteristic classes*, Ann. Math. Stud. **76**, Princeton Univ. Press, 1974.
- [15] Olmsted, J. M. H.: *Rational values of trigonometric functions*, Amer. Math. Monthly, **52** (1945), 507–508.
- [16] O’Neill, L.: *On the fixed point property for Grassmann manifolds*, Ph. D. Thesis, Ohio State University, Ohio, 1974.

- [17] Paranjape, K. H.—Srinivas, V.: *Self-maps of homogeneous spaces*, Invent. Math., **98** (1989), 425–444.
- [18] Ramani, V.—Sankaran, P.: *On degrees of maps between Grassmannians*, Proc. Ind. Acad. Sci. Math. Sci., **107** (1997), 13–19.
- [19] Sankaran, P.—Sarkar, S.: *Degrees of maps between Grassmann manifolds*, Osaka J. Math., **46** (2009), 1143–1161.
- [20] Sullivan, D.: *Infinitesimal computations in topology*, Publ. Math. IHES, **47** (1977), 269–331.
- [21] Sury, B.: *A curious polynomial identity*, Nieuw Arch. Wisk. (4), **11** (1993), 93–96.

* STAT-MATH UNIT

INDIAN STATISTICAL INSTITUTE

8TH MILE MYSORE ROAD

BANGALORE 560 059

INDIA.

E-mail address: `chakraborty.prateep@gmail.com`

** INSTITUTE OF MATHEMATICAL SCIENCES

IV CROSS ROAD, CIT CAMPUS

TARAMANI

CHENNAI 600113

INDIA.

E-mail address: `shreedevikm@imsc.res.in`, `masuti.shree@gmail.com`